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Random Surfaces with Two-Sided Constraints: An Application of the Theory of Dominant Ground States

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We consider some models of classical statistical mechanics which admit an investigation by means of the theory of dominant ground states. Our models are related to the Gibbs ensemble for the multidimensional SOS model with symmetric constraints $|\phi_x| \leqslant m/2$. The main result is that for $\beta \geqslant \beta_0$, where β_0 does not depend on m, the structure of thermodynamic phases in the model is determined by dominant ground states: for an even m a Gibbs state is unique and for an odd m the number of space-periodic pure Gibbs states is two.

KEY WORDS: Random surfaces; SOS model with symmetric constraints; dominant ground states.

1. INTRODUCTION: THE MODELS UNDER CONSIDERATION

The present work has arisen from attempts to understand the picture of low-temperature phase transitions in the following hard-core lattice model. Suppose that at sites of the ν-dimensional cubic lattice $\mathbf{Z}^\nu$ (the dimension ν is always supposed to be $\geqslant 2$) there are placed classical spins taking values $0,..., m$. A spin configuration is given by $\eta = \{\eta_x, x \in \mathbf{Z}^\nu\}$ with $\eta_x = 0,..., m$. The site–site interaction is nearest neighbor and is reduced to a hard-core threshold repulsion: the value $\eta_x + \eta_{x'}$ for any pair x, x' of nearest-neighbor sites should not exceed m. The statistical weight of a con-

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822/64/1-2-8

figuration[4] $\eta(V)$ in a finite volume $V \subset \mathbf{Z}^\nu$ is $z^{N(\eta(V))}$, with $z \geqslant 0$ a fugacity of the system and $N(V)$ the total sum $\sum_{x \in V} \eta_x$. The low-temperature regime corresponds here to large values of fugacity.

Such a model naturally appears in the theory of communication networks (see, e.g., ref. 1) and in the theory of neuron networks (see ref. 2). We think, however, that this model is interesting from the standpoint of statistical mechanics as well: it is formulated in simple terms and has some peculiar features.

Ground states of the model are close-packing configurations. These are simply chessboards and may be labeled by an index $k = -m/2, -m/2 + 1, \ldots, m/2$ related to values η_x taken at the points of "even" and "odd" sublattices:

$$\begin{aligned} \eta_x^{(k)} &= \frac{m}{2} + k, \qquad \text{if } x \text{ is even} \\ \eta_x^{(k)} &= \frac{m}{2} - k, \qquad \text{if } x \text{ is odd} \end{aligned} \tag{1.1}$$

[a point $x = (x^1, \ldots, x^\nu) \in \mathbf{Z}^\nu$ is called even if the sum $\sum_{i=1}^{\nu} |x^i|$ is even, and odd otherwise].

The analysis of the model shows that not all of those ground states create thermodynamic phases. The number of phases equals one for m even, whereas the number of space-periodic pure phases is two for m odd. This is established by means of the theory of dominant ground states (DGS) developed in refs. 3–5 (see also relevant references cited in these papers).

The model just considered may be treated as a "subensemble" of the well-known SOS model. In the SOS model, a configuration is determined as a function $\phi: \mathbf{Z}^\nu \to \mathbf{Z}^1$, or, equivalently, as a surface formed by horizontal and vertical ν-dimensional unit "plaquettes" in $\mathbf{Z}^{\nu+1}$. The Hamiltonian in a finite volume $V \subset \mathbf{Z}^\nu$ reads

$$H(\phi(V)) = \sum_{\langle x, x' \rangle \subset V} |\phi_x - \phi_{x'}| \tag{1.2}$$

The sum is extended to all (unordered) pairs $\langle x, x' \rangle$ of nearest-neighbor sites of $\mathbf{Z}^\nu$; ϕ_x denotes the value of ϕ at $x \in \mathbf{Z}^\nu$. We notice that due to the invariance property of the Hamiltonian (1.2) wrt shifts $\phi \mapsto \phi + \text{const}$, we can assume that our functions ϕ take values from a shifted lattice

[4] In speaking of configurations or surfaces (see below), we use sometimes the argument V to indicate the volume where they are considered. When it is clear (or immaterial) what volume is in mind, this argument is omitted.

$\mathbf{Z}^1+\text{const}$. To pass from a configuration η of the previous model to a surface ϕ, we use the following rule:

$$\phi_x=k, \qquad \text{if} \quad \eta_x=\eta_x^{(k)}, \quad x\in\mathbf{Z}_\nu \tag{1.3}$$

where $\eta_x^{(k)}$, $k=-m/2, \; -m/2+1,..., m/2$, is defined in (1.1). The equivalent formula is given by

$$\phi_x=\left(\frac{m}{2}-\eta_x\right)(-1)^{|x|}, \qquad x\in\mathbf{Z}^\nu \tag{1.3a}$$

and the inverse formula by

$$\eta_x=\frac{m}{2}-(-1)^{|x|}\,\phi_x, \qquad x\in\mathbf{Z}^\nu \tag{1.4}$$

For an even m the function ϕ defined by (1.3), (1.3a) takes integer and for an odd m half-integer values between $-m/2$ and $m/2$.

The ground states of the Hamiltonian (1.2) are simply constant surfaces $\phi^{(k)}$:

$$\phi_x^{(k)}=k, \qquad x\in\mathbf{Z}^\nu$$

It is easy to check that for η and ϕ related by (1.3)–(1.4) and for any k the following equality holds:

$$z^{N(\eta)-N(\eta^{(k)})}=\exp\{-\beta[H(\phi)-H(\phi^{(k)})]\}$$

with $\beta=-(1/2\nu)\ln z$. Notice that not every surface ϕ may be obtained in the rhs of the formulas (1.3), (1.3a). Namely, ϕ must necessarily obey the following two constraints. We first notice the aforementioned constraint: for any $x\in\mathbf{Z}^\nu$

$$|\phi_x|\leqslant\frac{m}{2} \tag{1.5}$$

Another constraint is that for any nearest-neighbor pair $\langle x,x'\rangle$ with an even x (and hence, with an odd x'),

$$\phi_x\geqslant\phi_{x'} \tag{1.6}$$

The last constraint leads to certain simply formulated geometrical restrictions which will be discussed later.

The results on the original model which have been briefly mentioned may be immediately reformulated in terms of the SOS model with the restrictions (1.5), (1.6). Namely, if β is large enough, then, for an even m, a unique phase of the model is created by the ground state $\phi^{(0)}$, and for an odd m there exist precisely two space-periodic pure phases which are created by the ground states $\phi^{(\pm 1/2)}$.

It turns out that the phase transition picture in the SOS model with the constraints (1.5), (1.6) is the same as in the model with only the one constraint (1.5). The last model was discussed in ref. 6, where such a picture has been conjectured (following an earlier paper[7]).

The present paper contains the proofs of the results just mentioned, for both models. For the sake of simplicity, we discuss in detail the case of the second model only: the change needed to cover the first one is outlined explicitly and may be easily worked out by the reader. It is worth noticing that our results are valid for $\beta \geqslant \beta_0$, where β_0 depends on v, but not on m.

As was said before, the method of proof is based on searching and investigating DGSs of a model under consideration. We study a so-called ensemble of low-energy excitations of a ground-state surface; this is the subject of Section 2. As a result, we conclude that the surfaces $\phi^{(0)}$ and $\phi^{(\pm 1/2)}$ which are the farthest ones from the bounding hyperplanes $\phi^{(\pm m/2)}$ minimize the free energy of such an ensemble. In principle, the results of Section 2 allow us to reduce the problem of proving the assertions stated to verifying conditions formulated in ref. 4. However, we have preferred to devote a separate part of the paper, Section 3, to a direct proof. This is motivated by two reasons. First, we control that all the constants involved are uniform in m. Second, we use a slightly different technique, which was proposed in ref. 8. We believe that this technique is sometimes more convenient. An example is a short proof of a so-called "strip" theorem[9] which is the extension of a result from ref. 10 to the cases under consideration.

2. THE MAIN RESULT AND THE BASIC ESTIMATE

As said in the previous section, we focus on the SOS model with the only constraint (1.5). Recall that for m even, a function ϕ takes integer values, and for m odd, half-integer ones. It is convenient to treat the value space as $\mathbf{Z}^1 + \kappa$, where κ is zero for an even m and one-half for an odd m. The main result is given by the following theorem.

Theorem 2.1. There exists a constant $\beta_0 = \beta_0(v) < \infty$ such that for any m and any $\beta \geqslant \beta_0$ the following assertions hold true:

(i) If m is even, then the model possesses a unique Gibbs state which is a small perturbation of the zero ground-state surface $\phi^{(0)}$.

(ii) If m is odd, then the set of space-periodic extremal Gibbs states contains precisely two elements, phases which are small perturbations of $\pm 1/2$ ground states $\phi^{(\pm 1/2)}$; those Gibbs states are transformed into one another by the symmetry mapping $\phi \mapsto -\phi$.

The same assertion holds for the SOS model with the two constraints (1.5) and (1.6).

As usual in the theory of DGSs, the crucial part in the proof of Theorem 2.1 is studying a gas of low-energy excitations around a given ground state and then comparing corresponding "reduced" partition functions for different ground states. This will enable us to find DGSs of the model under consideration. The comparison is performed in Lemma 2.2.

Geometrically, every surface $\phi(V)$: $V \mapsto (\mathbf{Z}^1 + \kappa)$ on a finite volume $V \subset \mathbf{Z}^\nu$ is composed of vertical "walls" and horizontal "ceilings." Horizontal ceilings are formed by unit horizontal ν-dimensional plaquettes with centers at points $(x, \phi_x) \in \mathbf{Z}^\nu \times (\mathbf{Z}^1 + \kappa)$, and vertical walls by unit vertical ν-dimensional plaquettes which are projected onto those $(\nu - 1)$-dimensional plaquettes of the dual lattice $\mathbf{Z}^\nu = (\mathbf{Z}^1 + 1/2)^\nu$ which separate nearest-neighbor points x, $x' \in \mathbf{Z}^\nu$ with $\phi_x \neq \phi_{x'}$.

Vertical walls are decomposed into "signed cylinders": each cylinder γ is determined by a contour $\tilde{\gamma} = \tilde{\gamma}(\gamma)$ (the base of the cylinder) and an integer $l = l(\gamma)$ (the signed length of the cylinder). A contour is defined, as usual, as a connected set of $(\nu - 1)$-dimensional plaquettes of the dual lattice $\tilde{\mathbf{Z}}^\nu$ such that, for any $(\nu - 2)$-dimensional plaquette of the contour, the number of $(\nu - 1)$-dimensional contour plaquettes passing through it is even. The sign of $l(\gamma)$ indicates in which of the two vertical directions a cylinder γ is "growing."

The exterior and interior of a cylinder are understood here in terms of the basic contour.

Given a surface ϕ, the corresponding collection of cylinders satisfies a consistency condition given below (we always have in mind "maximal" cylinders by forbidding two cylinders to have coinciding contours). As is readily seen, any consistent collection of cylinders determines, in a unique way, a surface in a volume with given boundary condition.

A collection of cylinders $\{\gamma\}$ is called consistent if the following two conditions are fulfilled. (I) For any pair γ_1, γ_2 of cylinders from the collection having the same sign, the interiors of the corresponding contours $\tilde{\gamma}_1, \tilde{\gamma}_2$ either lie at a mutual distance >1 or are embedded into one another (the contours $\tilde{\gamma}_1, \tilde{\gamma}_2$ may themselves have common plaquettes in this case). (II) For any pair γ_1, γ_2 of cylinders from the collection having opposite signs, the interiors of the contours $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ either do not intersect (i.e., lie at a mutual distance $\geqslant 1$) or are embedded into one another, and in this case the contours $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ do not intersect.

Associated with a cylinder $\gamma = (\tilde{\gamma}, l)$ is the statistical weight

$$w(\gamma) = \exp(-\beta \, \|\gamma\|) \tag{2.1}$$

where $\|\gamma\|$ is the number of vertical ν-dimensional plaquettes in γ:

$$\|\gamma\| = |\tilde{\gamma}| \; |l| \tag{2.2}$$

$|\tilde{\gamma}|$ denotes the number of $(\nu-1)$-dimensional plaquettes in the basic contour $\tilde{\gamma}$. Notice that for any surface ϕ in a finite volume $V \subset \mathbf{Z}^\nu$ its statistical weight reads

$$\exp[-\beta H(\phi(V))] = \prod_\gamma w(\gamma) \tag{2.3}$$

where the product is extended to cylinders from the corresponding consistent collection.

It is time to point out the difference between the two models under consideration. First, the model with the constraints (1.5) and (1.6) admits only contours where any pair of adjacent $(\nu-1)$-dimensional plaquettes is mutually orthogonal. Second, the definition of a cylinder must be modified for this model: this is now a quadruple $\gamma = (\tilde{\gamma}, l, \varepsilon, \varepsilon')$ where $\tilde{\gamma}$ and l are as before and $\varepsilon, \varepsilon' = \pm 1$ indicate the signs of an external and internal ground-state surface separated by the contour $\tilde{\gamma}$. This is caused by the fact that condition (1.6) does not admit cylinders $(\tilde{\gamma}, l, \varepsilon, \varepsilon')$ with $\varepsilon\varepsilon' = -1$ and with the property that $(-1)^{|x|} \neq \varepsilon$ for some (or, equivalently, for any) point $x \in \mathbf{Z}^\nu$ which adjoins a plaquette of the contour $\tilde{\gamma}$ from the exterior.

Therefore, the above condition of consistency should be completed with the following requirement. (III) For any pair γ_1, γ_2 of cylinders from a collection $\{\gamma\}$ such that the basic contours $\tilde{\gamma}_1, \tilde{\gamma}_2$ are not separated by any third contour $\tilde{\gamma}$ being the base of a cylinder from the same collection, (a) if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are mutually external (i.e., do not lie one inside the other), then $\varepsilon(\gamma_1) = \varepsilon(\gamma_2)$, and (b) if the interior of $\tilde{\gamma}_2$ contains that of $\tilde{\gamma}_1$, then $\varepsilon(\gamma_1) = \varepsilon'(\gamma_2)$, and if, on the contrary, the interior of $\tilde{\gamma}_1$ contains that of $\tilde{\gamma}_2$, then $\varepsilon(\gamma_2) = \varepsilon'(\gamma_1)$.

This difference between the two models exhibits itself every time when we need to estimate from above the number of contours $\tilde{\gamma}$ with a fixed value of $|\tilde{\gamma}|$ passing through a fixed plaquette. In both cases, however, this number is bounded by $\exp(c\,|\tilde{\gamma}|)$, where the constant c depends only on the dimension ν. Modulo this fact, the arguments in the proof of Theorem 2.1 are identical to those in the proof of the similar assertion for the model with the constraints (1.5), (1.6).

Let us fix now a finite volume $V \subset \mathbf{Z}^\nu$ and a boundary condition $\phi^{(k)}(V^c)$. We extract, from the Gibbs ensemble in V with this boundary condition, a reduced subensemble formed by (signed) cylinders γ with

$$\operatorname{diam} \tilde{\gamma} \leqslant 10 \frac{m}{2} 2\nu \tag{2.4}$$

(the number 10 plays here no particular role, of course). This reduced ensemble corresponds to a gas of low-energy excitations of the ground state $\phi^{(k)}$ in the volume V.

Let $Z^{(k)}(V)$ denote the partition function of the reduced ensemble in V with the boundary condition $\phi^{(k)}(V^c)$. Let $Z^{(-)}(V)$ denote the partition function $Z^{(0)}(V)$ for m even (when $\kappa = 0$) and any one of (equal) partition functions $Z^{(\pm 1/2)}(V)$ for m odd (when $\kappa = 1/2$). Our aim is to write down a (convergent) polymer expansion for $\log Z^{(k)}(V)$ and then to compare coefficients for different k values.

We notice that, according to (2.3), the partition functions $Z^{(k)}(V)$ admit the following cluster representation:

$$Z^{(k)}(V) = \sum_{\{\gamma\}} \prod_{\gamma} w(\gamma) \tag{2.5}$$

where the sum is extended to consistent collections of cylinders which obey the boundary condition.

In auxiliary assertions which follow we deal with various positive constants which do not depend on m, but depend on v. Those constants are denoted c, c', etc., though their values can change from one place to another. We also omit the sentence that β is $\geqslant \beta_0$, where $\beta_0 < \infty$ depends on v but not on m and k.

Lemma 2.2. For any m and any k with $0 < |k| \leqslant m/2$, the following bound holds true:

$$\frac{Z^{(k)}(V)}{Z^{(-)}(V)} < \exp\left[-c\,|V| \exp\left(-\beta \frac{m}{2} 2v\right)\right] \tag{2.6}$$

Here and below $|V|$ denotes the number of points $x \in V$.

Proof of Lemma 2.2. For definiteness, let us assume that m is even. First, consider the case where $|k| \geqslant m/3$. Notice that the partition functions $Z^{(k)}(V)$ and $Z^{(0)}(V)$ have a "common" part which is given by the contribution of those surfaces ϕ which obey $|\phi_x - k/2| \leqslant m/2 - |k|/2$ [for $Z^{(k)}(V)$] and $|\phi_x + k/2| \leqslant m/2 - |k|/2$ [for $Z^{(0)}(V)$], $x \in V$. These surfaces are transformed into one another by a vertical shift. The common part is denoted $Z^{(k,0)}(V)$.

To analyze distinct parts of our reduced partition functions $Z^{(k)}(V)$ and $Z^{(0)}(V)$, we introduce the notion of a wedding cake. This is a consistent collection π of cylinders of a fixed sign, which satisfies the following additional condition. The collection π contains a unique cylinder γ with maximally-external contours $\tilde{\gamma}$ [we denote them, respectively, $\gamma_e(\pi)$ and $\tilde{\gamma}_e(\pi)$], and for all maximally-internal contours $\tilde{\gamma}'$ from π, the sum $\sum_i l(\gamma_i)$

is the same [we denote it $l(\pi)$]. The sum is taken over the cylinders γ_i whose basic contours $\tilde{\gamma}_i$ are intersected by a path on $\mathbf{Z}^\nu$ connecting $\tilde{\gamma}_e(\pi)$ with $\tilde{\gamma}'$ (included).

In physical terms, all the "summits" (or "holes") of the surface representing a wedding cake (or, briefly, a cake) must be of the same "height" (or "depth"). The statistical weight of a wedding cake π reads

$$w(\pi) = \prod_{\gamma \in \pi} w(\gamma) \tag{2.7}$$

The product is taken over all cylinders γ from π.

In a similar way one can introduce a more general notion of a semi-cake: this is a consistent collection σ of cylinders, possibly, of the both signs, satisfying the aforementioned conditions (which mean uniqueness of the maximally-external basic contour and equality of the absolute altitudes of maximally-internal cylinders). We shall use for semi-cakes similar notations $\tilde{\gamma}_e(\sigma)$ and $l(\sigma)$. The statistical weight of a semi-cake is given by the formula identical to (2.7).

The concepts of a cake and a semi-cake are useful because, on one hand, they enable us to describe various possible ways for our surfaces to reach a given level, and on the other hand, both cakes and semi-cakes have a "summability" property, as shown by the following lemma. Let us call two semi-cakes congruent if they may be transformed into one another by a horizontal space shift and vertical reflection. Clearly, $w(\sigma_1) = w(\sigma_2)$ for any pair of congruent semi-cakes σ_1 and σ_2.

Lemma 2.3. Given $\bar{l}$, $\bar{d} > 0$, there exists $\beta_1 = \beta_1(\nu, \bar{d}/\bar{l}) < \infty$ such that for any $\beta \geqslant \beta_1$ the following bound holds true:

$$\sum w(\sigma) \leqslant \exp[-(\beta - c)\,\bar{l}2\nu] \tag{2.8}$$

The sum is taken here over all congruence classes of wedding cakes σ which satisfy the following two conditions: (i) diam $\tilde{\gamma}_e(\sigma) \leqslant \bar{d}$, (ii) $|l(\sigma)| \geqslant \bar{l}$.

Proof of Lemma 2.3. Due to our restrictions on the diameter diam $\tilde{\gamma}_e(\sigma)$ and the length $l(\sigma)$, it is possible to fix, in a standard way, a family of horizontal ν-dimensional plaquettes containing at most $\|\sigma\|\ \bar{d}/\bar{l}$ plaquettes which, together with the vertical plaquettes of the cake σ, form a connected plaquette hypersurface. Here $\|\sigma\|$ denotes the total number of the vertical plaquettes in σ. The number of different semi-cakes σ with a fixed value of $\|\sigma\|$ is growing exponentially with $\|\sigma\|$, and hence the sum in the lhs of (2.8) is bounded from above by a sum of a geometric progression.

We return now to the proof of Lemma 2.2. It is convenient to use the following

Proposition 2.4. The following bound holds true:

$$Z^{(k)}(V) \leqslant Z^{(k,0)}(V) \exp\left\{c' \, |V| \exp\left[-(\beta - c)\left(\frac{m}{2} + 1\right) 2\nu\right]\right\} \tag{2.9}$$

Proof of Proposition 2.4. Let us introduce a procedure of affiliating (gluing) a semi-cake σ to a surface ϕ. The necessary requirement is that the whole collection of cylinders γ from both σ and ϕ satisfies the consistency condition (see above). The result of affiliating is a new surface ϕ' such that its collection of cylinders is the union of the collections of σ and ϕ.

Notice that any surface ϕ contributing to $Z^{(k)}(V)$ may be viewed as a result of affiliating, to a surface $\tilde{\phi}$ which contributes to $Z^{(k,0)}(V)$, a family (possibly, empty) of semi-cakes $\{\sigma\}$ with $(m/2) + 1 \leqslant |l(\sigma)| \leqslant (m/2) + |k|$ and $-\text{sign}\, l(\sigma) = \text{sign}\, k$. To verify this, it is enough to define, for a surface ϕ contributing to $Z^{(k)}(V)$, a "reversible" procedure of "erasing" long semi-cakes from ϕ in such a way that the resulting surface will contribute to $Z^{(k,0)}(V)$. For example, such an erasing procedure may be defined as the iteration of the following two-step operation: (a) partitioning the surface into "maximal" semi-cakes, (b) deleting cylinders of those maximal semi-cakes which have the lengths with the absolute value exceeding $(m/2)$. The procedure is ended when the resulting surface will not contain anymore maximal cakes subjected to erasing (in which case the surface will contribute to $Z^{(k,0)}(V)$).

Each affiliating leads to multiplication of the statistical weight of the underlying surface $\tilde{\phi}$ by the product $\prod w(\sigma)$. Taking the sum over all ways to affiliate a family of wedding cakes to a given surface $\tilde{\phi}$ gives a factor which does not exceed

$$\left[1 + \sum w(\sigma)\right]^{|V|} \leqslant \exp\left\{c' \, |V| \exp\left[(\beta - c)\left(\frac{m}{2} + 1\right) 2\nu\right]\right\} \tag{2.10}$$

The sum $\sum w(\sigma)$ in the lhs is the same as in Lemma 2.3.

The next step in the proof of Lemma 2.2 is the following result.

Proposition 2.5. Let $\tilde{Z}^{(k)}(V)$ denote the part of the sum $Z^{(k)}(V)$ which contains the contributions of those surfaces $\tilde{\phi}$ for which $\#\{x \in V: \tilde{\phi}_x = k\} \leqslant |V|/2$. Then

$$\tilde{Z}^{(k)}(V) \leqslant \exp\left[-(\beta - c)\frac{|V|}{2}\right] \tag{2.11}$$

Proof of Proposition 2.5. For each surface $\tilde{\phi}$ contributing to $\tilde{Z}^{(k)}(V)$, the collection of its vertical cylinders may be completed to a connected set of ν-dimensional plaquettes by adding at most $|V|$ horizontal plaquettes. On the other hand, the same surface $\tilde{\phi}$ contains at least $|V|/2$ vertical plaquettes. The quantity $\tilde{Z}^{(k)}(V)$ does not exceed thereby the sum of a geometric progression

$$\sum_{n=|V|/2}^{\infty} \exp(-\beta n)(c')^{2n} \leqslant \exp\left[-(\beta - c)\frac{|V|}{2}\right] \tag{2.12}$$

n is the number of vertical plaquettes in $\tilde{\phi}$.

We now establish a lower bound for $Z^{(0)}(V)$:

Proposition 2.6. The following bound holds true:

$$Z^{(0)}(V) \geqslant Z^{(k,0)}(V) \exp\left[c\,|V| \exp\left(-\beta \frac{m}{6} 2\nu\right)\right] \tag{2.13}$$

Proof of Proposition 2.6. As follows from Proposition 2.5, for β large enough,

$$Z^{(k,0)}(V) - \tilde{Z}^{(k)}(V) \geqslant \tfrac{1}{2} Z^{(k,0)}(V)$$

Furthermore, after affiliating, to a surface $\tilde{\phi}$ contributing to the difference $Z^{(k,0)}(V) - \tilde{Z}^{(k)}(V)$, a collection of congruent cylinders γ with sign $\gamma = \text{sign}\ k$ which have, as a base, unit contours and have the length $m/2 - k + 1$, we still obtain a surface which contributes to $Z^{(0)}(V)$, but not to $\tilde{Z}^{(k)}$. The sum over all such collections is greater than

$$\{\exp[-\beta H(\tilde{\phi})]\}\left\{1 + \exp\left[-\beta 2\nu\left(\frac{m}{2} - k + 1\right)\right]\right\}^{|V|/2\nu}$$

$$\geqslant \{\exp[-\beta H(\tilde{\phi})]\} \exp\left[c'\,|V| \exp\left(-\beta \frac{m}{6} 2\nu\right)\right]$$

The assertion of Proposition 2.6 now follows immediately.

To finish the proof of Lemma 2.2 for $|k| \geqslant m/3$, it suffices to collect the results of Propositions 2.4–2.6:

$$\frac{Z^{(k)}(V)}{Z^{(0)}(V)} \leqslant \frac{Z^{(k,0)}(V) \exp\{c_1'\,|V| \exp[-(\beta - c_1)(m/2 + 1)\,2\nu]\}}{Z^{(k,0)}(V) \exp[c'\,|V| \exp(-\beta(m/6)\,2\nu)]}$$

$$\leqslant \exp\left[c\,|V| \exp\left(-\beta \frac{m}{2} 2\nu\right)\right] \tag{2.14}$$

Let us now turn to the case $0 < |k| < m/3$. We shall treat partition functions $Z^{(k)}(V)$, $|k| \leqslant m/3$ (including $k=0$), as small perturbations of the partition function $Z_{\infty}^{(0)}(V)$ corresponding to the standard SOS model [without the constraint (1.5)] with the zero boundary condition. For the standard SOS model, for any finite volume V one is able to write down a convergent (for $\beta \geqslant \beta_0$) polymer expansion for $\log Z_{\infty}^{(0)}(V)$ in terms of "polymers" composed of cylinders. As usual, by a polymer we mean here (and below) a "connected" collection of cylinders (or more complicated geometrical objects) which satisfy the condition: for any pair of cylinders γ, γ' from the collection, there exists a sequence of cylinders $\gamma_1,\ldots, \gamma_s$ from the same collection such that $\gamma_1 = \gamma$, $\gamma_s = \gamma'$ and γ_j is inconsistent with γ_{j+1} for any $j = 1,\ldots, s-1$.

Therefore, for writing a convergent polymer expansion for $\log Z^{(k)}(V)$, it suffices to expand the quantity

$$\log Z^{(k)}(V) - \log Z_{\infty}^{(0)}(V) = \log \frac{Z^{(k)}(V)}{Z_{\infty}^{(0)}(V)} \tag{2.15}$$

Furthermore, to write down a polymer expansion for (2.15), we have to write (and to investigate) a cluster representation for the ratio

$$\frac{Z^{(k)}(V)}{Z_{\infty}^{(0)}(V)} \tag{2.16}$$

We shall treat the ratio (2.16) as a probability $P_V(A_{k,m})$, where $P_V = P_{\infty,V}^{(0)}$ is the Gibbs measure for the standard SOS model in the volume V with the zero boundary condition, and the event $A_{k,m}$ is given by

$$A_{k,m} = \left\{\phi\colon \left|\phi_x - \frac{k}{2}\right| \leqslant \frac{m}{2} - \frac{|k|}{2},\ x \in V\right\} \tag{2.17}$$

Given a wedding cake π with $|l(\pi) + k/2| \leqslant m/2 - |k|/2 + 1$, it is convenient to introduce an event $A_{(\pi)}$. Namely, we say that a wedding cake π is external (on a surface ϕ where it lives) if, for any horizontal surface ψ ($\psi_x = \text{const}$, $x \in V$) which has a nonempty intersection with π, the contour $\tilde{\gamma}$ of a (unique) cylinder γ from π which intersects the surface ψ is an external one in the whole collection of contours being bases for those cylinders of the surface ϕ which have nonempty intersection with ψ. We set then

$$A_{(\pi)} = \{\phi\colon \text{the wedding cake } \pi \text{ is living on the surface } \phi \text{ and moreover, } \pi \text{ is external on } \phi\}$$

Denoting by χ_π the indicator function of A_π, we can write, in conventional notations,

$$P_V(A_{k,m}) = P_V\left(\prod_{\tilde{\pi}} (1-\chi_{\tilde{\pi}})\right) \tag{2.18}$$

where the product is taken over all possible wedding cakes $\tilde{\pi}$ over V with $|l(\tilde{\pi}) + k/2| = m/2 - |k|/2 + 1$ and with a unique maximally-internal contour. After expanding the product $\prod_{\tilde{\pi}}$, we arrive at the quantity

$$P_V\left(\sum_{\{\tilde{\pi}\}} (-1)^{\sharp\{\tilde{\pi}\}} \prod_{\tilde{\pi}} \chi_{\tilde{\pi}}\right)$$

where the sum is extended to collections $\{\tilde{\pi}\}$ of cakes which satisfy the aforementioned conditions and are consistent in the following sense. If we take the whole family of cylinders from all the cakes of our collection, then this family satisfies the consistency condition (see above).

Such a collection $\{\tilde{\pi}\}$ forms a surface which is composed of all the cylinders involved, and we can partition this surface into pairwise "disjoint" wedding cakes π, but already without the condition of uniqueness of a maximally-internal contour. Notice, however, that any such cake π may be decomposed, in a unique way, into a union of cakes having only one maximaly-internal contour. This allows us to write

$$P_V\left(\sum_{\{\tilde{\pi}\}} (-1)^{\sharp\{\tilde{\pi}\}} \prod_{\tilde{\pi}} \chi_{\tilde{\pi}}\right) = \sum_{\{\pi\}} (-1)^{\sharp_i\{\pi\}} P_V(\chi_{\{\pi\}}) \tag{2.19}$$

the sum in the rhs of (2.19) is taken over all collections of pairwise disjoint wedding cakes π with $|l(\pi) + k/2| = m/2 - |k|/2 + 1$, $\sharp_i\{\pi\}$ denotes the total number of maximally-internal contours in the collection $\{\pi\}$, and $\chi_{\{\pi\}}$ is the indicator of the corresponding surface.

We shall treat the rhs of (2.19) as a "partition function" for a model of interacting wedding cakes. We can use again the polymer expansion for the standard SOS model and write

$$P_V(\chi_{\{\pi\}}) = \prod_{\pi} P_{\mathbf{Z}^\nu}(\pi)\, Q_V(\{\pi\}) \tag{2.20}$$

where $\log Q_V(\{\pi\})$ is the sum of a convergent polymer series arising from the expansion for $\log Z_\infty^{(0)}(V)$. Notice that each polymer contributing to $\log Q_V(\{\pi\})$ binds at least two wedding cakes from $\{\pi\}$.

The polymer series for $\log Q_V(\{\pi\})$ is of the form

$$\log Q_V(\{\pi\}) = \sum_q \mathbf{w}(q) \tag{2.21}$$

where q is a polymer of cylinders. We notice that the polymers q are from the standard SOS model, and the weight $\mathbf{w}(q)$ admits the bound

$$|\mathbf{w}(q)| \leqslant \exp[-(\beta - c)\, \|q\|] \tag{2.22}$$

where

$$\|q\| = \sum_{\gamma \in q} \|\gamma\| \tag{2.23}$$

and $\|\gamma\|$ is given by (2.2).

The "canonic" formula

$$\exp[\mathbf{w}(q)] = 1 + \tilde{\mathbf{w}}(q)$$

allows us to obtain a cluster representation

$$P_V(A_{k,m}) = \sum_{\{\Gamma\}} \prod_{\Gamma} \mathbf{W}(\Gamma) \tag{2.24}$$

Here Γ is a "connected multicake," i.e., a collection of consistent wedding cakes which are connected by a collection of polymers q, and the weight $\mathbf{W}(\Gamma)$ reads

$$\mathbf{W}(\Gamma) = \prod_{\pi \in \Gamma} P_{\mathbf{Z}^\nu}(\pi) \prod_{q \in \Gamma} \tilde{\mathbf{w}}(q)(-1)^{\#_i\{\pi\}} \tag{2.25}$$

[see (2.19), (2.20)]. We notice that any one-cake Γ contains no polymer, and hence in this case,

$$\mathbf{W}(\Gamma) = P_{\mathbf{Z}^\nu}(\pi)(-1)^{\#_i\{\pi\}} \tag{2.26}$$

Due to the imposed restriction $|k| < m/3$ and the condition (2.4), we get the bound

$$\frac{|l(\pi)|}{\operatorname{diam} \bar{\gamma}_e(\pi)} \geqslant \left(\frac{m}{2} - |k| + 1\right)(10m\nu)^{-1}$$
$$\geqslant \left(\frac{m}{6} + 1\right)(10m\nu)^{-1} \geqslant \frac{1}{60\nu}$$

Therefore, we can apply Lemma 2.3 ensuring that the bound (2.8) holds for the statistical weight $w(\pi)$ [see (2.7)]. The next remark is that $P_{\mathbf{Z}^\nu}(\pi)$ does not exceed $w(\pi)$. Hence, the same bound holds for $P_{\mathbf{Z}^\nu}(\pi)$, too:

$$\sum_{\pi} P_{\mathbf{Z}^\nu}(\pi) \leqslant \exp\left[-(\beta - c)\left(\frac{m}{6} + 1\right) 2\nu\right] \tag{2.27}$$

The sum in the lhs is extended to the same objects as in (2.8).

The estimates (2.8) and (2.27) enable us to pass to a standard polymer expansion for

$$\log P_V(A_{k,m}) = \log Z^{(k)}(V) - \log Z^{(0)}_{\infty}(V) \tag{2.28}$$

It is convenient to extract, from the whole polymer sum which gives the value $\log P_V(A_{k,m})$, a quantity $\Delta_{V,k,m}$. The quantity $\Delta_{V,k,m}$ is defined as the sum of the contributions of those polymers which include precisely one connected multicake which contains just a single cake which has a unique maximally-internal contour which, finally, is a unit one (i.e., the boundary of a unit plaquette of $\mathbf{Z}^\nu$). These "elementary" polymers are obviously identified with wedding cakes π having the properties just listed and are denoted in the sequel as $[\pi]$.

Notice once more that all the constructions performed so far are valid for $k=0$ as well. This is also the case of Proposition 2.7 below.

Proposition 2.7. The following bound holds true:

$$|\log P_V(A_{k,m}) - \Delta_{V,k,m}| \leqslant |V|\, e^{-(\beta-c)(m/2)(4\nu-1)} \tag{2.29}$$

Proof of Proposition 2.7. The contribution of those polymers which include more than one connected multicake does not exceed

$$|jV| \exp\left[-2(\beta-c)\frac{m}{2}2\nu\right] \tag{2.30}$$

This follows from the fact that (a) the statistical weight of such a polymer contains at least two factors $P_{\mathbf{Z}^\nu}(\pi_j)$ corresponding to cakes π_j, and any such factor is $\leqslant \exp[-\beta(m/2)\,2\nu]$, and (b) the number of polymers of a given size grows exponentially with the size [this leads to the presence of the constant c in (2.30)].

Next, the contribution of those polymers which include just one connected multicake but contain more than one cake is estimated again by (2.30) due to the same kind of argument.

Further, consider the contribution of those polymers which include just one connected multicake containing a single cake which has more than one maximally-internal contour. The statistical weight of any such polymer is bounded from above by $\exp[-2\beta(m/2)\,2\nu]$, and their contribution is, as before, not greater than (2.30).

Finally, we have to analyze the contribution of those polymers which are reduced to one wedding cake with a unique maximally-internal contour which, however, is not unit (i.e., is not reduced to the boundary of a unit plaquette of $\mathbf{Z}^\nu$). The statistical weight of any such polymer does not exceed $\exp[-\beta(m/2)(4\nu-1)]$. The bound (2.29) now follows immediately.

We can now write

$$\log Z^{(k)}(V) - \log Z^{(0)}(V)$$
$$= \log Z^{(k)}(V) - \log Z_{\infty}^{(0)}(V) + \log Z_{\infty}^{(0)}(V) - \log Z^{(0)}(V) \quad (2.31)$$

Due to Proposition 2.7,

$$\log Z^{(k)}(V) - \log Z_{\infty}^{(0)}(V) = \varDelta_{V,k,m} + \delta_k \quad (2.32)$$

$$-\log Z_{\infty}^{(0)}(V) + \log Z^{(0)}(V) = \varDelta_{V,0,m} + \delta_0 \quad (2.33)$$

where

$$|\delta_0| + |\delta_k| \leqslant |V| \, e^{-(\beta - c)(m/2)(4\nu - 1)} \quad (2.34)$$

The final step in the proof of Lemma 2.2 for $0 < |k| < m/3$ is to compare the quantities $\varDelta_{V,k,m}$ and $\varDelta_{V,0,m}$.

Proposition 2.8. The following bound holds true:

$$\varDelta_{V,k,m} - \varDelta_{V,0,m} \leqslant -\tfrac{1}{2} |V| \, e^{-\beta(m/2)2\nu} \quad (2.35)$$

Proof of Proposition 2.8. By definition,

$$\varDelta_{V,k,m} = -\sum_{[\pi]} P_{\mathbf{Z}^\nu}(\pi) \quad (2.36)$$

where $|l(\pi) + k| = m/2 + 1$. A similar formula (with k replaced by zero) holds for $\varDelta_{V,0,m}$.

We see that for any polymer $[\pi]$ (or, equivalently, wedding cake π) contributing either to $\varDelta_{V,k,m}$ or to $\varDelta_{V,0,m}$, the absolute value $|l(\pi)|$ can take three values only: $m/2 - |k| + 1$, $m/2 + 1$, and $m/2 + |k| + 1$. For any wedding cake π with $|l(\pi)| = m/2 - |k| + 1$, three associated cakes, $\pi_1 = \pi_1(\pi)$ and $\pi_2(\pi)$ with $|l(\pi_1)| = |l(\pi_2)| = m/2 + 1$ and $\pi_3 = \pi_3(\pi)$ with $|l(\pi)| = m/2 + |k| + 1$, give contributions either to $\varDelta_{V,k,m}$ or to $\varDelta_{V,0,m}$: π and π_3 contribute to $\varDelta_{V,k,m}$ and π_1 and π_2 to $\varDelta_{V,0,m}$. Geometrically, all the π_i are obtained from π after increasing the absolute value of the length of the maximally-internal cylinder: in the case of π_1 and π_2 by $|k|$, and in the case of π_3 by $2|k|$. Therefore,

$$P_{\mathbf{Z}^\nu}(\pi_1) = P_{\mathbf{Z}^\nu}(\pi_2) \leqslant P_{\mathbf{Z}^\nu}(\pi) \, e^{-\beta|k|2\nu} \quad (2.37)$$

$$P_{\mathbf{Z}^\nu}(\pi_3) \leqslant P_{\mathbf{Z}^\nu}(\pi) \, e^{-2\beta|k|2\nu} \quad (2.38)$$

Now,

$$\Delta_{V,k,m} - \Delta_{V,0,m} = \sum_{[\pi]:|l(\pi)| = m/2 - |k| + 1} [-P_{\mathbf{Z}^\nu}(\pi) + P_{\mathbf{Z}^\nu}(\pi_1) + P_{\mathbf{Z}^\nu}(\pi_2) - P_{\mathbf{Z}^\nu}(\pi_3)]$$
$$\leqslant -\frac{1}{2} \sum_{[\pi]:|l(\pi)| = m/2 - |k| + 1} P_{\mathbf{Z}^\nu}(\pi) \tag{2.39}$$

The rhs of (2.39) will increase if we reduce the summation to those wedding cakes π for which the maximally-external contour $\tilde{\gamma}_e(\pi)$ is merely a unit one (in which case it will be in fact the unique contour of π). For any such cake π

$$P_{\mathbf{Z}^\nu}(\pi) \geqslant \tfrac{1}{2} e^{-\beta(m/2 - |k| + 1)2\nu}$$

This leads to the upper bound

$$\Delta_{V,k,m} - \Delta_{V,0,m} \leqslant -\tfrac{1}{4} |V| \, e^{-\beta(m/2 - |k| + 1)2\nu} \tag{2.40}$$

which finishes the proof of Proposition 2.8.

Bounds (2.34) and (2.40) complete the proof of Lemma 2.2 for the case of an even m. The case of an odd m is considered in a similar way.

3. DERIVATION OF THEOREM 2.1 FROM THE BASIC ESTIMATE

As mentioned before, the proof of Theorem 2.1 which follows presents a concrete version of a theory of DGSs. We want to point out once more that, in all the steps of our proof, the bounds used are uniform in m.

As before, we start with the case of an even m. The proof is naturally divided into two (interconnected) parts: the proof of existence of Gibbs states indicated in Theorem 2.1 and that of uniqueness thereof. Let us begin with the proof of existence.

Consider the partition function $\Xi(V)$ of our model in a volume V with the zero boundary condition $\phi^{(0)}(V^c)$. We can write the following representation:

$$\Xi(V) = \sum_{\psi(V)} \sum_{\phi(V)}^{(\psi(V))} \exp[-\beta H(\psi(V))] \times \exp\{-\beta[H(\phi(V)) - H(\psi(V))]\} \tag{3.1}$$

The external sum is taken here over the surfaces $\psi(V)$ with the boundary condition $\phi^{(0)}(V^c)$ such that for any vertical cylinder γ of $\psi(V)$ the estimate opposite to (2.4) is fulfilled,

$$\operatorname{diam} \tilde{\gamma} > 10 \frac{m}{2} 2\nu \tag{3.2}$$

Such a surface ψ is called in the sequel a factor-surface. The internal sum $\sum^{(\psi(V))}$ is extended to surfaces ϕ which: (a) contain all the cylinders of ψ, and (b) can contain, in addition, other cylinders γ which obey (2.4).

Let us now introduce the notion of a factor-contour. Given a factor surface ψ, we extract connected components $S\,[= S(\psi)]$ of the set $\{x \in V: \psi_x \neq 0\}$ and consider the restrictions of ψ to those connected components. A single restriction is called a factor-contour of a factor-surface ψ and denoted $\Omega\,[= \Omega(\psi)]$ and the corresponding connected component $S\,[= S(\Omega)]$ of the set $\{x \in V: \psi_x \neq 0\}$ is called the support of Ω.

As a "subsurface" of a surface ψ, a factor-contour Ω is composed of its vertical cylinders $\gamma_i = \gamma_i(\Omega)$ which satisfy the consistency condition (see above). Notice that, among cylinders forming a factor-contour surface Ω, there is a unique maximally-external one, $\gamma_e(\Omega)$, for which the basic contour $\tilde{\gamma}_e(\Omega)$ is the maximally-external one among the contours being the bases for cylinders of Ω. On the other hand, there can exist (in general, several) maximally-internal cylinders, $\gamma^{(j)}_{\rm int}(\Omega)$, whose basic contours, $\tilde{\gamma}^{(j)}_{\rm int}(\Omega)$, separate $S(\Omega)$ from its complement [there can exist as well maximally-internal cylinders without this property, whose basic contours lie inside $S(\Omega)$].

The partition function Ξ admits the representation

$$\Xi(V) = \sum_{\psi(V)} Z(V, \psi) \exp[-\beta H(\psi(V))] \tag{3.3}$$

where

$$Z(V, \psi) = \prod_{k=-m/2}^{m/2} Z^{(\psi)}(V_k) \tag{3.4}$$

$V^{(k)} = V^{(k)}(\psi)$ denotes the set $\{x \in V: \psi_x = k\}$ and $Z^{(\psi)}(V^{(k)})$ is the partition function in the volume $V^{(k)}$ with the boundary condition induced by the factor surface ψ.

As usual, to prove the existence of a Gibbs state generated by the zero boundary condition, we have to construct the polymer expansion for $\log \Xi(V)$. In fact, it is sufficient to construct a polymer expansion for the factor model, i.e., to expand the quantity

$$\log\left(\frac{\Xi(V)}{Z^{(0)}(V)}\right) = \log \Xi(V) - \log Z^{(0)}(V) \tag{3.5}$$

because $\log Z^{(0)}(V)$ was expanded in the course of proving Lemma 2.2 for $|j| < m/3$.

The quantity (3.5) may be written as

$$\log \left\{ \sum_{\psi(V)} \frac{Z(V, \psi)}{Z^{(0)}(V)} \exp[-\beta(H(\psi(V))]\right\} \tag{3.6}$$

or, equivalently, in the factor-contour form,

$$\log \sum_{\Omega} \prod_{\Omega} w(\Omega)\, Q(\{\Omega\}) \tag{3.7}$$

The sum is taken here over all collections of consistent factor-contours [the consistency means simply that the supports $S(\Omega)$ are pairwise disjoint]. Further, the statistical weight $w(\Omega)$ of a factor-contour Ω reads

$$w(\Omega) = \exp[-\beta H(\Omega)] \frac{\prod_k Z^{(k)}(S^{(k)}(\Omega))}{Z^{(0)}(S(\Omega))} \tag{3.8}$$

and the quantity $Q(\{\Omega\})$ gives the interaction between the factor-contours Ω:

$$Q(\{\Omega\}) = \frac{\prod_\Omega Z^{(0)}(S(\Omega))\, Z^{(0)}(V \backslash (\bigcup_\Omega S(\Omega)))}{Z^{(0)}(V)} \tag{3.9}$$

The domains $S^{(k)}(\Omega)$ in the rhs of (3.8) are defined in the same way as $V^{(k)}(\psi)$: $S^{(k)}(\Omega) = \{x \in S(\Omega)\colon \Omega_x = k\}$. The term $\exp[-\beta H(\Omega)]$ may be written in the following obvious form:

$$\exp[-\beta H(\Omega)] = \prod_{\gamma(\Omega)} w(\gamma(\Omega)) \tag{3.10}$$

where the product is taken over all cylinders $\gamma(\Omega)$ of the factor-contour Ω and the statistical weight $w(\gamma)$ is defined in (2.1). Notice, for further use, that, by construction, cylinders $\gamma(\Omega)$ satisfy the condition (3.2).

The quantity $\log Q(\{\Omega\})$ admits a convergent polymer expansion [of a similar form as the expansion (2.21)]. More precisely, we have in mind the polymer expansion for $Z^{(0)}(V)$ which was constructed in the proof of Lemma 2.2 for $|k| < m/3$. This enables us to write the representation of the "partition function"

$$\frac{\Xi(V)}{Z^{(0)}(V)} = \sum_{\{\Omega\}} \prod_{\Omega} w(\Omega)\, Q(\{\Omega\}) \tag{3.11}$$

in terms of noninteracting (= nonintersecting) clusters of factor contours. It is a standard (but tremendous) construction (see, e.g., refs. 11 and 12) and we omit the details from this paper (notice that a similar approach, namely, the multicake construction, was already used in Section 2).

We now have to pass from the aforementioned cluster factor-contour representation for (3.9) to the convergent polymer expansion for $\log[\Xi(V)/Z^{(0)}(V)]$. The crucial point here is the following result.

Lemma 3.1. The following bound holds true:

$$\sum_{\Omega:\tilde{\gamma}_e(\Omega)\ni 0} w(\Omega)\exp[|S(\Omega)|\exp\left(-5\beta\frac{m}{2}2\nu\right)\leqslant\exp\left(-5\beta\frac{m}{2}2\nu\right) \tag{3.12}$$

The sum in (3.12) is extended to those factor-contours Ω for which the maximally-external contour $\tilde{\gamma}_e(\Omega)$ passes through the origin of the dual lattice $\tilde{\mathbf{Z}}^\nu$.

To derive the convergence of the polymer expansion under consideration from Lemma 3.1, it suffices to use the general criterion from ref. 13. Furthermore, convergence of the polymer expansion for $\log[\Xi(V)/Z^{(0)}(V)]$ implies, in the standard way, existence of the limit Gibbs state under discussion. Hence, the problem of existence is reduced to proving Lemma 3.1.

Proof of Lemma 3.1. The statistical weight $w(\Omega)$ satisfies the bound

$$w(\Omega)\leqslant\prod_{\gamma(\Omega)} w(\gamma(\Omega))\exp\left[-c\,|S(\Omega)|\exp\left(-\beta\frac{m}{2}2\nu\right)\right] \tag{3.13}$$

The product is taken here over all cylinders of a factor-contour Ω and $w(\gamma)$ is given, as before, by (2.2). The bound (3.13) follows immediately from Lemma 2.2 and the definition (3.8).

The problem of proving (3.12) is reduced to proving the bound: for any fixed cylinder γ

$$\sum_{\Omega:\gamma_e(\Omega)=\gamma} w(\Omega)\exp\left[|S(\Omega)|\exp\left(-\beta 5\frac{m}{2}2\nu\right)\right]\leqslant w(\gamma) \tag{3.14}$$

The sum in the lhs of (3.12) is taken over all factor-contours Ω for which the maximally-external cylinder $\gamma_e(\Omega)$ coincides with γ. In fact, the sum (3.12) equals

$$\begin{aligned}
&\sum_{\gamma:\tilde{\gamma}\ni 0}\ \sum_{\Omega:\gamma_e(\Omega)=\gamma} w(\Omega)\exp\left[|S(\Omega)|\exp\left(-\beta 5\frac{m}{2}2\nu\right)\right]\\
&\quad\leqslant\sum_{\gamma:\tilde{\gamma}\ni 0} w(\Omega)\leqslant\sum_{n=10m\nu}^{\infty}\exp[(-\beta-c')\,n]\\
&\quad\leqslant\exp[(-\beta-c)\,10m\nu]\leqslant\exp(-\beta 5m\nu)
\end{aligned}$$

Let us now check (3.14). By using (3.13), we have to prove that

$$\sum_{\Omega:\gamma_e(\Omega)=\gamma} \prod_{\gamma(\Omega)} w(\gamma(\Omega)) \exp\left[-\frac{1}{2}|S(\Omega)| \exp\left(-\beta \frac{m}{2} 2v\right)\right] \leqslant w(\gamma) \tag{3.15}$$

To estimate the sum in the lhs of (3.15), we fix a family of maximally-internal cylinders $\{\gamma^{(j)}\}$ and consider the sum

$$\begin{aligned}&\sum_{\Omega:\gamma_e(\Omega)=\gamma, \{\gamma^{(j)}_{\text{int}}(\Omega)\}=\{\gamma^{(j)}\}} \prod_{\gamma(\Omega)} w(\gamma(\Omega)) \exp\left[-\frac{c}{2}|S(\Omega)| \exp(-\beta m v)\right] \\ &\quad = \exp\left[-\frac{c}{2}|S(\tilde{\gamma}, \{\tilde{\gamma}^{(j)}\})| \exp(-\beta m v)\right] w(\gamma) \prod_j w(\gamma^{(j)}) \\ &\quad \times \sum_{\Omega:\gamma_e(\Omega)=\gamma, \{\gamma^{(j)}_{\text{int}}(\Omega)\}=\{\gamma^{(j)}\}} \prod_{\gamma(\Omega)}^{*} w(\gamma(\Omega))\end{aligned} \tag{3.16}$$

where $S(\tilde{\gamma}, \{\tilde{\gamma}^{(j)}\})$ is the domain (on the lattice $\mathbf{Z}^{\nu}$) between the maximally-external contour $\tilde{\gamma}$ and the maximally-internal contours $\tilde{\gamma}^{(j)}$, and the product $\prod^{*}$ is taken over all cylinders of a factor-surface Ω which are different from "extremal" cylinders γ and $\gamma^{(j)}$. Since the bases of all these cylinders satisfy the restriction (3.2) and are situated inside the support $S(\tilde{\gamma}, \{\tilde{\gamma}^{(j)}\})$, the following bound holds true:

$$\begin{aligned}&\sum_{\Omega:\gamma_e(\Omega)=\gamma, \{\gamma^{(j)}_{\text{int}}(\Omega)\}=\{\gamma^{(j)}\}} \prod_{\gamma(\Omega)}^{*} w(\gamma(\Omega)) \\ &\quad \leqslant \exp\left[\frac{c}{4}|S(\tilde{\gamma}, \{\tilde{\gamma}^{(j)}\})| \exp(-\beta m v)\right]\end{aligned} \tag{3.17}$$

Next, we perform the summation, for a fixed cylinder γ, over all cylinders $\gamma^{(j)}$. We use the so-called Zahradnik trick (see ref. 14). Given a positive d, consider an auxiliary partition function $Z'_d(V)$ corresponding to the ensemble of surfaces with the zero boundary condition and with the restriction that for any cylinder γ of our surface its basic contour $\tilde{\gamma}$ obeys diam $\tilde{\gamma} > d$:

$$Z'_d(V) = \sum_{\{\gamma\}} \prod_{\gamma} w(\gamma) \tag{3.18}$$

The sum in the rhs of (3.18) is extended to consistent collections of cylinders with the aforementioned property.

Proposition 3.2.[5] For any finite volume $V' \subset \mathbf{Z}^{\nu}$ the following bound holds true:

$$\sum_{\{\gamma\}^{(e)}} \frac{\prod_{\gamma} Z'_d(S(\tilde{\gamma}))\, w(\gamma)}{Z'_d(V')} \leqslant 1 \tag{3.19}$$

[5] Proposition 3.2 is a particular case of a more general assertion from ref. 8.

Here, $S(\tilde{\gamma})$ is the interior of a contour $\tilde{\gamma}$ (which is the base for a cylinder γ) and the sum in the rhs of (3.19) is taken over collections $\{\gamma\}^{(e)}$ of mutually-external cylinders inside the volume V'.

The proof of Proposition 3.2 is immediate: as follows from the definition (3.18) and from the restrictions determining the domain of summation in (3.19),

$$\sum_{\{\gamma\}^{(e)}} \prod_{\gamma} Z'_d(S(\tilde{\gamma}))\, w(\gamma) \leqslant Z'_d(V')$$

Indeed, the restrictions defining the sum in the rhs are more liberal than those in the lhs.

Returning to the proof of Lemma 3.1, we can write, in light of Proposition 3.2,

$$\begin{aligned}\sum_{\{\gamma^{(j)}\}} \exp\left[-\frac{c}{4}|S(\tilde{\gamma}, \{\tilde{\gamma}^{(j)}\})| \exp(-\beta m v)\right] \prod_j w(\gamma^{(j)})\\ \leqslant \sum_{\{\gamma^{(j)}\}} \frac{\prod_j Z'_d(S(\tilde{\gamma}^{(j)}))\, w(\gamma^{(j)})}{Z'_d(S(\tilde{\gamma}))} \leqslant 1\end{aligned} \tag{3.20}$$

The sum $\sum_{\{\gamma^{(j)}\}}$ is performed here over collections of maximally-internal cylinders whose contours form the internal boundary of $S(\Omega)$, and the value d in the rhs of (3.20) is taken to be equal to $10mv$.

Substituting (3.20) and (3.17) in (3.16), we arrive at the bound (3.15). This finishes the proof of existence of the limit Gibbs state with the zero boundary condition.

Let us now turn to the question of uniqueness. The uniqueness of the limit Gibbs state which was constructed before follows in the standard way from Lemma 3.3 below. Consider the Gibbs ensemble of surfaces, in a volume V which is now supposed to be a cube with center at the origin and with edges parallel to coordinate axes, and with an arbitrary boundary condition $\phi(V^c)$. Given an $r > 1$, denote by $B_{V,r}$ the event that there exists a collection of contours $\{\tilde{\gamma}^{(j)}\}$ satisfying the twofold conditions: (i) the contours $\tilde{\gamma}^{(j)}$ encircle a volume $V_{(r)} \subset V$ such that $|V \setminus V_{(r)}| \geqslant 2rvl(V)^{v-1}$ [by $l(V)$ we denote here and below the length of the edge of the cube V], and (ii) $\phi_x = 0$ for any j and any point x from the interior of the contour $\tilde{\gamma}^{(j)}$ which adjoins $\tilde{\gamma}^{(j)}$.

Recall that the value of β is supposed to be large enough, but does not depend on m as well as all the constants appearing in the course of the proof.

Lemma 3.3. For any $r \geqslant c\beta e^{\beta(m/2)2v}$ and any (even) m, the following relation holds:

$$\lim_{l(V)\to\infty} \sup_{\phi(V^c)} P_V^{(\phi)}(\mathbf{C}B_{V,r}) = 0 \tag{3.21}$$

where $P_V^{(\phi)}$ denotes the probability distribution of the Gibbs ensemble under consideration and $\mathbf{C}B_{V,r}$ is the complement of the event $B_{V,r}$.

Proof of Lemma 3.3. Dealing with arbitrary boundary conditions, it is convenient to introduce the notion of a "boundary picture." By this we mean the whole family Υ_b of the factor contours Ω with the supports $S(\Omega)$ adjoining the boundary ∂V. We shall treat this family as a sort of factor-contour with the maximally-external contour $\tilde{\gamma}_e(\Upsilon_b)=\partial V$. We should notice that certain contours $\tilde{\gamma}^{(j)}_{\text{int}}(\Upsilon_b)$ separating the support $S(\Upsilon_b)=\bigcup_{\Omega\in\Upsilon_b} S(\Omega)$ from the set $V^{(0)}(\phi)=\{x\in V: \phi_x=0\}$ can adjoin the boundary ∂V. Further, the part of ∂V which attaches the zero portion of the boundary condition *must* be adjoined by some contours $\tilde{\gamma}^{(j)}_{\text{int}}(\Upsilon_b)$. Moreover, certain contours $\tilde{\gamma}^{(j)}_{\text{int}}(\Upsilon_b)$ may appear partially "glued" so that some of their plaquettes are counted twice and the interiors of those contours fail to be connected.

Dealing with the Gibbs ensemble corresponding to the boundary condition $\phi(V^c)$, we must attribute to a boundary picture Υ_b the statistical weight

$$w(\Upsilon_b)=[w(\gamma_e(\Upsilon b))]^{-1}\prod_{\gamma(\Upsilon_b):\gamma(\Upsilon_b)\neq\gamma_e(\Upsilon_b)} w(\gamma(\Upsilon_b))\times\frac{\prod_k Z^{(k)}(S^{(k)}(\Upsilon_b))}{Z^{(0)}(S(\Upsilon_b))}\tag{3.22}$$

[cf. the definition (3.8) of the statistical weight for a "standard" (zero-boundary-condition) contour].

We notice that the event $\mathbf{C}B_{V,r}$ means the following property of a boundary picture:

$$|S(\Upsilon_b)|=\sum_{\Omega\in\Upsilon_b}|S(\Omega)|\geqslant 2rv[l(V)]^{v-1}$$

Let us now estimate from above the probability

$$P_V^{(\phi)}(\{|S(\Upsilon_b)|\geqslant 2rv[l(V)]^{v-1}\})\tag{3.23}$$

It does not exceed

$$\sum_{\Upsilon_b:|S(\Upsilon_b)|\geqslant 2rv[l(V)]^{v-1}}[w(\gamma_e(\Upsilon_b))]^{-1}\times\prod_{\gamma(\Upsilon_b):\tilde{\gamma}(\Upsilon_b)\neq\partial V} w(\gamma(\Upsilon_b))\exp\left[-c\,|S(\Upsilon_b)|\exp\left(-\beta\frac{m}{2}2v\right)\right]\tag{3.24}$$

This follows from the following three facts. (a) The probability of a boundary picture Υ_b does not exceed its statistical weight (3.22). (b) The statistical weight of a boundary picture Υ_b under a boundary condition

$\phi(V^c)$ is obtained from its statistical weight under the zero boundary condition $\phi^{(0)}(V^c)$ by multiplication by $[w(\gamma_e(\Upsilon_b))]^{-2}$ (since, for a nonzero boundary condition, those plaquettes which are projected on the boundary ∂V do not give any contribution to the statistical weight of Υ_b). (c) By virtue of Lemma 2.2, the quantity

$$\prod_{\gamma(\Upsilon_b)} w(\gamma(\Upsilon_b)) \exp\left[-c\,|S(\Upsilon_b)| \exp\left(-\beta \frac{m}{2} 2\nu\right)\right]$$

gives an upper bound for the statistical weight of a boundary picture Υ_b under the zero boundary condition.

Now notice that the sum (3.24) does not exceed, for $r \geqslant c\beta e^{\beta(m/2)2\nu}$,

$$\exp\{2\nu[l(V)]^{\nu-1}\beta\} \exp\left\{-\frac{4}{c}\nu[l(V)]^{\nu-1} r \exp\left(-\beta\frac{m}{2}2\nu\right)\right\}$$
$$\leqslant \exp\{-2\nu[l(V)]^{\nu-1}\beta\} \tag{3.25}$$

Indeed, the first factor in the lhs of (3.25) is merely

$$\max[w(\gamma_e(\Upsilon_b))]^{-1} \exp(\beta\,|\partial V|)$$

As to the second factor, it is the upper bound for

$$\max_{|S(\Upsilon_b)| \geqslant 2r\nu[l(V)]^{\nu-1}} \exp\left[-\frac{2}{c}|S(\Upsilon_b)| \exp\left(-\beta\frac{m}{2}2\nu\right)\right]$$
$$\times \sum_{\Upsilon_b:\tilde{\gamma}_e(\Upsilon_b)=\partial V,\ |S(\Upsilon_b)| \geqslant 2r\nu[l(V)]^{\nu-1}} \ \prod_{\gamma(\Upsilon_b):\ \tilde{\gamma}(\Upsilon_b) \neq \partial V} w(\gamma(\Upsilon_b))$$
$$\times \exp\left[-\frac{2}{c}|S(\Upsilon_b)| \exp\left(-\beta\frac{m}{2}2\nu\right)\right] \tag{3.26}$$

The key remark here is that the last sum is $\geqslant 1$; this may be checked in the same way as in the course of deriving (3.20). This finishes the proof of Lemma 3.3.

Therefore, all the assertions of Theorem 2.1 related to an even m are proven.

The analysis of the structure of the space-periodic pure Gibbs states for the case of an odd m follows, again in the standard way, from Lemma 3.4 below, which is a counterpart of Lemma 3.3 for the case under consideration. Keeping in mind the same assumptions on a volume V as before, we denote now by $B_{V,r}$ the event that there exists a collection of contours $\{\tilde{\gamma}^{(j)}\}$ satisfying the above conditions (i) and (ii) with only the following change: instead of $\phi_x = 0$ in (ii), we require that $\phi_x = \pm 1/2$.

Lemma 3.4. The assertion of Lemma 3.3 holds in the situation of an odd m with the change just outlined.

The proof of Lemma 3.4 repeats that of Lemma 3.4 and we omit it.

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